

SINGULARITY AVOIDANCE BY COLLAPSING SHELLS IN QUANTUM GRAVITY¹

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Abstract

We discuss a model describing exactly a thin spherically symmetric shell of matter with zero rest mass. We derive the reduced formulation of this system in which the variables are embeddings, their conjugate momenta, and Dirac observables. A non-perturbative quantum theory of this model is then constructed, leading to a unitary dynamics. As a consequence of unitarity, the classical singularity is fully avoided in the quantum theory.

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The construction of a full, non-perturbative, quantum theory of gravity is one of the main open problems in physics. To achieve this goal, one is basically confronted with two options. On the one hand, one can attempt to construct first a full quantum theory of gravity and then derive interesting physical consequences. Superstring theory and the Ashtekar approach to canonical quantum gravity constitute two examples. On the other hand, one can try to construct first models which can be exactly quantized and which can therefore serve as a guide towards the full, unknown, theory. Such models can even have a direct bearing on concrete physical situations. This would be reminiscent of the first calculations of atomic spectra which were feasible without the knowledge of full quantum electrodynamics.

In our essay we shall follow the second route. The model that we shall discuss is a spherically-symmetric thin shell consisting of particles with zero rest mass (“lightlike shell”). According to the classical theory of general relativity, such a shell can collapse to form a black hole. Alternatively, it can emerge from a white hole and expand to infinity. The latter possibility is usually excluded for thermodynamical reasons.

The classical theory predicts that a genuine gravitational collapse leads to space-time singularities [1]. A special feature is the occurrence of a horizon during the collapse, a region from within no information can escape to the outside. It is a general expectation that a quantum theory of gravity can cure this situation, i.e., can lead to a singularity-free geometry. In fact, we shall show in this essay that a quantum theory for the lightlike shell leading to a singularity-free situation can be rigorously constructed. This will be a consequence of the unitary dynamics.

For the discussion of the model, we shall employ the approach of reduced quantization. In this approach, the variables can be neatly separated into pure gauge degrees of freedom (so-called embedding variables), their canonical momenta, and physical degrees of freedom [2]. The general existence of this “Kuchař decomposition” was shown in [3] by making a transformation to the usual (ADM) phase space of general relativity. In the construction, the notion of a background manifold plays a crucial role because one must define points by the choice of coordinates on some fixed manifold (a priori, spacetime points have no intrinsic meaning because they can be moved around by the diffeomorphism group).

The example of the lightlike shell constitutes the first application of this method [4]. For the coordinates on the fixed manifold, double-null coordinates U and V are chosen on the background manifold $\mathcal{M} = \mathbf{R}_+ \times \mathbf{R}$ (being effectively two-dimensional due to spherical symmetry). In these coordinates (which will play the role of the embedding variables), the metric has the form

$$ds^2 = -A(U, V)dUdV + R^2(U, V)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) . \quad (1)$$

From the demand that the metric be regular at the center and continuous at the

shell, the coefficients A and R are uniquely defined for any physical situation defined by the variables M (the energy of the shell), η (being $+1$ for an outgoing shell and -1 for an ingoing shell), and w (the location of the shell, where $w = u$ for the outgoing and $w = v$ for the ingoing case). The Penrose diagram for the outgoing shell is shown in Figure 1. It is important to note that the background manifold possesses a unique asymptotic region with \mathcal{J}^- defined by $U \rightarrow -\infty$ and \mathcal{J}^+ by $V \rightarrow +\infty$.

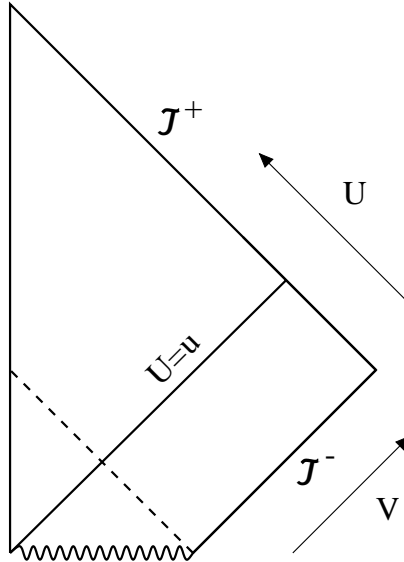


Figure 1: Penrose diagram for the outgoing shell in the classical theory. The shell is at $U = u$.

We shall first transform the classical theory of the shell into the formulation corresponding to the Kuchař decomposition and then construct the quantum theory. The standard (ADM) formulation of the shell was studied in [5]. One can perform an explicit transformation of these ADM variables into the new variables u and v , their momenta p_u and p_v , the embedding variables $U(\rho)$, $V(\rho)$, and their momenta [4]. The result is the action

$$S = \int d\tau (p_u \dot{u} + p_v \dot{v} - n p_u p_v) + \int d\tau \int_0^\infty d\rho (P_U \dot{U} + P_V \dot{V} - H) , \quad (2)$$

where $H = N^U P_U + N^V P_V$, and n , $N^U(\rho)$, and $N^V(\rho)$ are Lagrange multipliers. The first term in (2) contains the physical variables, while the second term contains the gauge variables. Observe that the Poisson algebra of the chosen set of observables p_u and u for $\eta = +1$ as well as p_v and v for $\eta = -1$ is gauge invariant in spite of the fact that it has been obtained by a calculation based on a gauge choice (the double-null coordinates U and V). This implies that our construction of the quantum

mechanics will also be gauge invariant. A crucial point is that the new phase space has non-trivial boundaries:

$$p_u \leq 0, \quad p_v \leq 0, \quad \frac{-u+v}{2} > 0. \quad (3)$$

The boundary defined by the last inequality is due to the classical singularity.

The system has now been brought into a form where it can be subject to quantization [6]. The restrictions (3) suggest the use of the so-called group-quantization method [7]. This method leads automatically to self-adjoint operators for the observables. A complete system of Dirac observables is given by p_u , p_v , as well as up_u and vp_v . They thus commute with the constraint $p_u p_v$. The Hilbert space is constructed from complex functions $\psi_u(p)$ and $\psi_v(p)$, where $p \in [0, \infty)$. The scalar product is defined by

$$(\psi_u, \phi_u) := \int_0^\infty \frac{dp}{p} \psi_u^*(p) \phi_u(p) \quad (4)$$

(similarly for $\psi_v(p)$). To handle the inequalities (3) it is useful to perform the following canonical transformation:

$$t = (u+v)/2, \quad r = (-u+v)/2, \quad (5)$$

$$p_t = p_u + p_v, \quad p_r = -p_u + p_v. \quad (6)$$

Upon quantization, one obtains the operator $-\hat{p}_t$ which is self-adjoint and has a positive spectrum. It is the generator of time evolution and corresponds to the energy operator \hat{M} . Since r is not a Dirac observable, it cannot directly be transformed into a quantum observable. It turns out that the following construction is useful [6]:

$$\hat{r}^2 := -\sqrt{p} \frac{d^2}{dp^2} \frac{1}{\sqrt{p}}. \quad (7)$$

This is essentially a Laplacian and corresponds to a concrete choice of factor ordering. It is a symmetric operator which can be extended to a self-adjoint operator. In this process, one is naturally led to the following eigenfunctions of \hat{r}^2 :

$$\psi(r, p) := \sqrt{\frac{2p}{\pi}} \sin rp, \quad r \geq 0. \quad (8)$$

One can also construct an operator $\hat{\eta}$ that classically would correspond to the direction of motion of the shell.

Now the basic formalism of the quantum theory is set and one can start to study concrete physical applications. We want to describe the dynamics of the shell by the evolution of a narrow wave packet. We take for $t = 0$ the following family of wave packets:

$$\psi_{\kappa\lambda}(p) := \frac{(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} p^{\kappa+1/2} e^{-\lambda p}, \quad (9)$$

where κ is a positive integer, and λ is a positive number with dimension of length. By an appropriate choice of these constants one can prescribe the expectation value of the energy and its variation. A sufficiently narrow wave packet can thus be constructed.

Since the time evolution of the packet is generated by $-\hat{p}_t$, one finds

$$\psi_{\kappa\lambda}(t, p) = \psi_{\kappa\lambda}(p)e^{-ipt} . \quad (10)$$

More interesting is the evolution of the wave packet in the r -representation. This is obtained by the integral transform (4) of $\psi_{\kappa\lambda}(t, p)$ with respect to the eigenfunctions (8). It leads to the exact result

$$\Psi_{\kappa\lambda}(t, r) = \frac{1}{\sqrt{2\pi}} \frac{\kappa!(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} \left[\frac{i}{(\lambda + it + ir)^{\kappa+1}} - \frac{i}{(\lambda + it - ir)^{\kappa+1}} \right] . \quad (11)$$

One interesting consequence can be immediately drawn:

$$\lim_{r \rightarrow 0} \Psi_{\kappa\lambda}(t, r) = 0 . \quad (12)$$

This means that the probability to find the shell at vanishing radius is zero! In this sense the singularity is avoided in the quantum theory. We emphasize that this is not a consequence of a certain boundary condition – it is a consequence of the *unitary evolution*. If the wave function vanishes at $r = 0$ for $t \rightarrow -\infty$ (asymptotic condition of ingoing shell), it will continue to vanish at $r = 0$ for all times. It follows from (11) that the quantum shell bounces and re-expands. Hence, no absolute event horizon can form, in contrast to the classical theory. However, an object that is locally similar to a black hole is not excluded by our results. In this way, the observational support for black holes is not contradicted.

Most interestingly, an essential part of the wave packet can even be squeezed below the expectation value of its Schwarzschild radius. This is achieved if the expectation value of the energy fulfills the condition

$$\langle \hat{M} \rangle > \frac{\lambda M_P}{\sqrt{2\pi}} M_P , \quad (13)$$

where M_P denotes the Planck mass, and $\lambda M_P \gg 1$ holds [6]. The wave packet can thus be squeezed below its Schwarzschild radius if its energy is much bigger than the Planck energy – a genuine quantum effect.

How can this behavior be understood? The unitary dynamics ensures that the ingoing quantum shell develops into a *superposition* of ingoing and outgoing shell if the region is reached where in the classical theory a singularity would form. In other words, the singularity is avoided by destructive interference in the quantum theory.

This is similar to the quantum-cosmological example of [8] where a superposition of a black hole with a white hole leads to a singularity-free quantum universe. Also here, the horizon becomes a superposition of “black hole” and “white hole” – its “grey” nature can be characterized by the expectation value of the operator $\hat{\eta}$ (a black-hole horizon would correspond to the value -1 and a white-hole horizon to the value +1). We emphasize that in this scenario no information-loss paradox would ever arise if such a behavior occurred for all collapsing matter (which sounds reasonable). In the same way, the principle of cosmic censorship would be implemented, since no naked singularities (in fact, no singularities at all) would form.

To summarize, the non-perturbative study of the above example demonstrates how the process of gravitational collapse may be viewed in quantum gravity. Whether the full, elusive, theory will be in accordance with this picture is of course an open question.

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